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Parity considerations in Andrews–Gordon identities

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ABSTRACT

In 1974, Andrews discovered the generating function for the partitions of n considered in a theorem due to Gordon. In a more recent paper, he reconsidered this generating function and gave refinements where additional restrictions involving parities are imposed. A combinatorial construction for the partitions enumerated by the mentioned generating function is given. Some of the Andrews' refinements are proven combinatorially, and a conjecture of his is settled.

1. Introduction

A partition λ of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_k > 0$ such that $n = \lambda_1 + \cdots + \lambda_k$ [5, Ch. 1]. One may impose some constraints such as requiring distinct parts, parts that belong to certain residue classes modulo some positive integer, and so on. Many theorems in partition theory assert the equinumerity of partitions of a given integer satisfying a condition and the partitions of that integer satisfying some other condition. Some theorems can be proven using generating functions. Others are proven using purely combinatorial methods and a following problem in such cases is to provide generating functions for partitions described in those results.

In 1961, Gordon proved that the number of partitions of n into parts that are not congruent to 0, $\pm a$ modulo $2k+1$ equals the number of partitions of n in which pairs of consecutive integers appear at most $k-1$ times and 1 appears at most $a-1$ times [12]. It is easy to write the generating function for the former sort of partitions, as they are given by

$$\prod_{\substack{n \geq 1 \\ n \not\equiv 0, \pm a \pmod{2k+1}}} \frac{1}{(1-q^n)}. \tag{1.1}$$

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However, a generating function was not given for partitions subject to the latter constraint unless $k = 2$. Gordon used purely combinatorial methods, and his proof is a generalization of Schur's [14] combinatorial proof of the Rogers–Ramanujan Identities. In the case $k = 2$, Gordon's result is a combinatorial interpretation of the famous Rogers–Ramanujan identities.

In 1966, Andrews [1] found that the following series is a solution to functional equations which derive from recurrences satisfied by $b_{k,a}(m, n)$, the number of partitions of n into m parts such that pair of consecutive integers occurs at most $k - 1$ times together and 1 occurs at most $a - 1$ times.

$$Q_{k,i}(x; q) = \sum_{n \geq 0} \frac{(-1)^n x^{kn} q^{(2k+1)n(n+1)/2-in}(1 - x^i q^{(2n+1)i})}{(q)_n (xq^{n+1})_\infty}. \tag{1.2}$$

This series previously appeared in [13] and [15]. Here,

$$(a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),$$

and

$$(a)_\infty = (a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$

In this sense,

$$\sum_{m,n \geq 0} b_{k,a}(m, n) x^m q^n = Q_{k,i}(x; q). \tag{1.3}$$

Upon substituting $x = 1$ and using Jacobi's Triple Product identity [5, Eq. (2.2.10)], Gordon's theorem follows.

In 1974, Andrews, in his Proceedings of the National Academy of Sciences paper [2, Eq. (2.5)], discovered the generating function for $b_{k,a}(m, n)$ as

$$\sum_{m,n \geq 0} b_{k,a}(m, n) x^m q^n = \sum_{n_1, \ldots, n_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}} x^{N_1 + \cdots + N_{k-1}}}{(q)_{n_1} \cdots (q)_{n_{k-1}}}, \tag{1.4}$$

where

$$N_r = n_r + \cdots + n_{k-1}.$$

He used the same functional equations for which $Q_{k,i}(x; q)$ is a solution, and established the right-hand side of (1.4) as another. Here, the exponent of q is the number being partitioned (n), and the exponent of x is the number of parts (m).

For $x = 1$, together with (1.3), and Jacobi's Triple Product Identity, it follows that

$$\sum_{n_1, \ldots, n_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}}}{(q)_{n_1} \cdots (q)_{n_{k-1}}} = \prod_{\substack{n \geq 1 \\ n \not\equiv 0, \pm a (\mathrm{mod}\ 2k+1)}} \frac{1}{(1 - q^n)}, \tag{1.5}$$

which has since been called the Andrews–Gordon Identities. Note that the proof does not use Gordon's original result.

The next problem was to provide a constructive explanation to explain Andrews' series (1.4), since it was not at all clear how the aforementioned partitions are generated. In 1980, Bressoud gave a combinatorial argument using ordinary partitions [9, Section 5] which allowed an extension to (1.4) with one more parameter. Bressoud's construction is inductive. There are other approaches which explain (1.4) such as Durfee dissection [4], multipartitions [6], or lattice paths [11].

In Section 2, the *Gordon marking* of a partition is defined, and a set of attributes to a partition is described. Backward and forward moves are defined which are restrictions of adding one or subtracting one from some part in the partition. This way, it is possible to keep some of those attributes invariant. The invariants are then related to the indices in the denominators of the generating function (1.4) thus arriving at a new combinatorial interpretation.

In a recent paper [7], Andrews revisited his generating function (1.4) and extended his results by considering some additional restrictions involving parities. He achieved those generalizations by using double recursions satisfied by $b_{k,a}(m, n)$ where additional constraints are imposed. This in turn gave larger sets of functional equations the solutions of which are variants of (1.4). Andrews then left the combinatorial explanations of the resulting generating functions as open problems. He made a conjecture, and gave a list of open problems. The method employed in Section 3 very naturally generalizes to explain most generating functions in [7] and proves Andrews' conjecture as well.

2. Background

We begin with a few definitions from [7].

Definition 2.1. Let $k \geq 2$, $k \geq a \geq 1$. $b_{k,a}(m, n)$ denotes the number of partitions of n into m parts such that 1 appears at most $a - 1$ times, and any pair of consecutive integers together appears at most $k - 1$ times.

Definition 2.2. Let $k \geq 2$, $k \geq a \geq 1$. $w_{k,a}(m, n)$ denotes the number of partitions of n enumerated by $b_{k,a}(m, n)$ such that even parts appear an even number of times.

Definition 2.3. Let $k \geq 2$, $k \geq a \geq 1$. $\overline{w}_{k,a}(m, n)$ denotes the number of partitions of n enumerated by $b_{k,a}(m, n)$ such that odd parts appear an even number of times.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n, i.e. $n = |\lambda| = \lambda_1 + \cdots + \lambda_m$.

Definition 2.4. The *Gordon marking* of a partition λ is an assignment of positive integers (marks) to λ such that parts equal to any given integer a are assigned distinct marks from the set $\mathbb{Z}_{>0} \setminus \left\{ r \,|\, \exists\, r\text{-marked } \lambda_j = a - 1 \right\}$ such that the smallest possible marks are used first. Let $\lambda^{(r)}$ denote the sub-partition of λ that consists of all r-marked parts. Let N_r be the number of r-marked parts (i.e. the number of parts in $\lambda^{(r)}$), and let $n_r = N_r - N_{r+1}$ for any positive integer r.

For instance, if $\lambda = (18, 17, 16, 15, 15, 13, 13, 11, 9, 7, 6, 6, 5, 4, 3, 2, 2)$, $(|\lambda| = 162)$, then its Gordon marking would be

$$\lambda = (18_2, 17_1, 16_3, 15_2, 15_1, 13_2, 13_1, 11_1, 9_1, 7_2, 6_3, 6_1, 5_2, 4_1, 3_3, 2_2, 2_1).$$

In fact, we can represent the Gordon marking by an array where the column indicates the value of a part, and the row (counted from bottom to top) indicates the mark, so the Gordon marking of λ above would be

$$\lambda = \left\{ \begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}.$$

We will use this representation throughout the paper.

There are several things to note here. First of all, Gordon marking is unique. For if a is the smallest part appearing in the partition, then there is a unique way to mark parts that are equal to a, and then there is a unique way to mark parts that are equal to $(a + 1)$ (if any), and so on.

$\lambda^{(r)}$ are sub-partitions with distinct non-consecutive parts, because no consecutive parts are assigned the same mark by definition. Also, for any r-marked λ_j, $r > 1$, there is a unique $(r - 1)$-marked $\lambda_{j_0} = \lambda_j$ or $\lambda_{j_0} = \lambda_j - 1$. This implies $N_1 \geq N_2 \geq \cdots$, and hence $n_1, n_2, \ldots \geq 0$.

Finally, if λ is enumerated by $b_{k,a}(m, n)$, then there are no k or greater marked parts, since each consecutive pair of integers together occur at most $(k - 1)$ times. In this case, we can restrict our attention on $N_1, \ldots, N_{k-1}$, and $n_1, \ldots, n_{k-1}$.

Definition 2.5. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a Gordon marked partition. Let $\lambda_j \neq 1$ be an r-marked part such that

(a) There are no $(r+1)$ or greater marked parts that are equal to λ_j or λ_j+1.
(b) There is an $r_0 \leq r$ such that there is an r_0-marked $\lambda_{j_0}=\lambda_j$, but no r_0-marked parts that are equal to λ_j-2.

Choose the smallest r_0 described in (b), and a *backward move of rth kind* on λ_j is replacing r_0-marked λ_{j_0} with an r_0-marked $\lambda_{j_0}-1$, and hence $|\lambda| \to |\lambda|-1$.

For instance,

$$\lambda = \left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\},$$

$\downarrow$ after a backward move of 3rd kind on 3-marked 6 becomes

$$\lambda' = \left\{\begin{array}{ccccccccccccc} & 3 & & \mathbf{5} & & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

Note that a backward move of 2nd kind is not possible for 2-marked 2, since (a) does not hold. Similarly, a backward move of the 1st kind on 1-marked 11 is not possible, since (b) fails.

We claim that a backward move of the rth kind preserves the marking of other unchanged parts. By (b) and the original marking, there are no r_0-marked parts that are equal to $\lambda_{j_0}-1$ or $\lambda_{j_0}-2$. It follows that there are no $r_0, r_0+1, \ldots$-marked parts that are equal to $\lambda_{j_0}-1$, since by the marking, any part equal to $\lambda_{j_0}-1$ that requires a higher mark than r_0 would be assigned r_0. Then, again by the marking, there are $r_0, r_0+1, \ldots, r$-marked parts equal to λ_j. By a similar reasoning, there are no $r_0, r_0+1, \ldots$-marked parts equal to λ_j+1, since any part that requires a higher mark than r_0 would be assigned $(r+1)$. That is ruled out by (a). This justifies the claim. Therefore, $N_1, N_2, \ldots$ are invariant under backward moves of any kind when conditions exist.

Definition 2.6. Let $\lambda=(\lambda_1,\ldots,\lambda_m)$ be a Gordon marked partition. Let λ_j be an r-marked part such that

(c) There are no $(r+1)$ or higher marked part equal to λ_j or λ_j+1,

and either

(d1) There is an r_0 marked part $\lambda_{j_0}=\lambda_j-1$, $r_0<r$ such that there are no r_0-marked parts equal to λ_j+1, and r_0+1 or higher marked parts equal to λ_j-1,

or

(d2) there are $1,\ldots,r-1$ marked parts equal to λ_j or λ_j+1, and no r-marked parts equal to λ_j+2.

A *forward move of* the *rth kind* is replacing r_0-marked λ_{j_0} with an r_0-marked $\lambda_{j_0}+1$ if (c) and (d1) holds; and replacing r-marked λ_j with an r-marked λ_j+1 if (c) and (d2) holds, and (d1) fails; hence $|\lambda| \to |\lambda|+1$.

For example,

$$\lambda = \left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\},$$

$\downarrow$ after a forward move of 3rd kind on 3-marked 16 becomes

$$\lambda'' = \left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & & \mathbf{16} & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

$\downarrow$ After another forward move of 3rd kind on 3-marked 16, it becomes

$$\lambda''' = \left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & & \mathbf{17} & \\ 2 & & & 5 & & 7 & & & 13 & & 16 & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

Observe that a forward move of the 2nd kind is not possible on 2-marked 13 in λ, since neither (d1) nor (d2) holds. Similarly, a forward move of the 2nd kind on 2-marked 2 is not possible, since (c) fails. To be more precise, we can replace that 2-marked 2 with a 2-marked 3 as some forward move, but that would be a forward move of the 3rd kind for 3-marked 3.

We claim here also that a forward move of the rth kind preserves the Gordon marking of unchanged parts. For when (c) and (d1) holds, there are no $r_0 + 1$ or higher marked parts that are equal to $\lambda_j - 1$ so that the deletion of r_0-marked $\lambda_{j_0} (= \lambda_j - 1)$ would spare the mark r_0 for them. Also, there are no r_0 marked parts equal to λ_j, but there are $r_0, r_0 + 1, \ldots, r$-marked parts equal to λ_j by the Gordon marking. Once r_0-marked λ_{j_0} is deleted, however, we need to alter the marking of parts equal to λ_j. That is avoided by the introduced r_0-marked $\lambda_{j_0} + 1 = \lambda_j$. By (c) and (d1), there are no r_0 or greater marked parts equal to $\lambda_j + 1$, therefore the marking of the other parts is not affected.

When (c) and (d2) holds, but (d1) does not, then there are $1, \ldots, (r - 1)$-marked parts equal to either λ_j or $\lambda_j + 1$, so an extra $\lambda_j + 1$ would be assigned mark r after the deletion of r-marked λ_j. Also, similar to the reasoning following the definition of a backward move of the rth kind, there are no $r+1$ or higher marked parts equal to $\lambda_j + 2$. So in this case as well, the Gordon marking of the other parts is unaltered, and hence $N_1, N_2, \ldots$ are invariant.

Note that in the above example for forward moves, the first move is possible in virtue of (c) and (d1), and the second move is possible in virtue of (c) and (d2).

Proposition 2.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a Gordon marked partition. Let $\lambda_j \neq 1$ be an r-marked part. If conditions exist for a backward move of the rth kind on λ_j, then conditions will exist for a forward move of the rth kind on the same part after the backward move is performed. Conversely, if conditions exist for a forward move of the rth kind on λ_j, then conditions will exist for a backward move of the rth kind on the same part after the forward move is performed.*

Moreover, the moves made in given orders will fix λ.

In other words, so many forward and that many backward moves, or vice versa, on the same part are inverse transformations on λ when conditions exist for the first sequence of moves.

Remark. 'Same part' refers to λ_j if another strictly smaller marked part is altered, and to $\lambda_j \pm 1$ if λ_j itself is altered.

Proof. Assume that a backward move of the rth kind is performed on λ_j, and an r_0-marked λ_{j_0} is replaced by an r_0-marked $\lambda_{j_0} - 1$.

If $r_0 < r$, then the forward move is to be performed on λ_j. In this case, (a) implies (c), and by the arguments following Definition 2.5, (d1) holds for the above $r_0 < r$, so the r_0-marked $\lambda_{j_0} - 1$ is replaced back by r_0-marked $\lambda_{j_0} - 1 + 1 = \lambda_{j_0}$.

Otherwise if $r_0 = r$, then the forward move is to be performed on r-marked $\lambda_j - 1$. (a) and the arguments following Definition 2.5 imply (c) under the substitution $\lambda_j \to \lambda_j - 1$. By (b), $r_0 = r$ and (d1) fails. By the Gordon marking of the original λ (d2) holds, and r-marked $\lambda_j - 1$ is replaced back by r-marked $\lambda_j - 1 + 1 = \lambda_j$. In either case, λ is fixed, so the first claim is proven.

For the second claim, if (c) and (d1) held, then (a) holds. There cannot be any r_0 or higher marked part equal to $\lambda_j + 1$, because there were no r_0 marked parts equal to λ_j before the forward move. Also, for some $r_0 < r$, (b) holds by the Gordon marking of original λ. r_0 in (d1) will be the smallest such for (b), since for any smaller $r_1 < r_0$, there are r_1-marked parts that are equal to either $\lambda_j - 1$ or $\lambda_j - 2$, by the marking. In this case, r_0-marked λ_{j_0} will first be replaced by r_0-marked $\lambda_{j_0} + 1$, and then replaced back again by r_0-marked $\lambda_{j_0} + 1 - 1 = \lambda_{j_0}$.

Otherwise if (c) and (d2) held, but (d1) failed, then by the Gordon marking and (c), there are no r or higher marked parts equal to $\lambda_j + 1$ or $\lambda_j + 2$, so (a) holds upon substituting $\lambda_j \to \lambda_j + 1$. Also, when $r = r_0$, then by the Gordon marking, there are r_1-marked parts ($r_1 < r$) equal to either λ_j or $\lambda_j - 1$ and by (d2), for all r_1 for which there is an r_1-marked part equal to $\lambda_j - 1$, there is also an r_1-marked part equal to $\lambda_j + 1$. Conversely, for any $r_1 < r$, the presence of an r_1-marked part equal to $\lambda_j + 1$ together with r-marked λ_j in the initial Gordon marked partition forces the presence of an r_1-marked part equal to $\lambda_j - 1$, by the Gordon marking. Thus, (b) holds only for $r_0 = r$, and for no smaller mark.

The r-marked $\lambda_j + 1$ is replaced back by r-marked $\lambda_j + 1 - 1 = \lambda_j$. In either case, λ is fixed. This justifies the second claim and concludes the proof. □

As an example, please note that a forward move of the 3rd kind on 3-marked 5 in λ' above gives us λ back. So does a backward move of the 3rd kind on 3-marked 16 in λ''.

Proposition 2.8. *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a Gordon marked partition. Let $\lambda_{j_1} < \lambda_{j_2}$ be two r-marked parts such that there are no r or higher marked λ_{j_3} for which $\lambda_{j_1} < \lambda_{j_3} < \lambda_{j_2}$.*

(i) *If conditions exist for a backward move of the rth kind on λ_{j_1}, and (a) is satisfied for λ_{j_2}, then the move made on λ_{j_1} will enable a backward move of the rth kind on λ_{j_2}.*

(ii) *If conditions exist for a forward move of the rth kind on λ_{j_2}, (c) is satisfied for λ_{j_1}, and either there are no $r + 1$ or higher marked parts equal to $\lambda_{j_1} - 1$, or for all $r_0 < r$ there are r_0 marked parts equal to λ_{j_1}, then the move made on λ_{j_2} will enable a forward move of the rth kind on λ_{j_1}.*

Proof. (i) When (b) fails for λ_{j_2}, then for all $r_0 \leq r$, there is an r_0 marked part equal to $\lambda_{j_2} - 2$ whenever there is an r_0 marked part equal to λ_{j_2}. In particular, $\lambda_{j_1} = \lambda_{j_2} - 2$. Once a backward move of the rth kind is made on λ_{j_1}, an $r_0 \leq r$ will be spared to satisfy (b) for λ_{j_2}.

(ii) (c) is satisfied both for λ_{j_1} and for λ_{j_2}.

For the first possibility, when both (d1) and (d2) fail for λ_{j_1}, then by (d2) there is an r-marked part equal to $\lambda_{j_1} + 2$. That is, $\lambda_{j_2} = \lambda_{j_1} + 2$. Once the move is performed on λ_{j_2}, if r-marked λ_{j_2} is replaced by r-marked $\lambda_{j_2} + 1$, (d2) will be satisfied for λ_{j_1} thanks to (d2) for λ_{j_2}. Or, some $r_0 < r$ will be spared after replacing an r_0 marked part equal to $\lambda_{j_2} - 1 = \lambda_{j_1} + 1$. By the remarks following the definitions of backward and forward moves, the same r_0 will satisfy (d1) for λ_{j_2}, since there are no $r_0 + 1$ or higher marked parts equal to $\lambda_{j_2} - 1 = \lambda_{j_1} + 1$.

For the second possibility, i.e. λ having $1, \ldots, r$-marked parts equal to λ_{j_1}, the only hindrance is that there are $1, \ldots, r$-marked parts equal to $\lambda_{j_1} + 2$, again, $\lambda_{j_2} = \lambda_{j_1} + 2$. In this case, the only possible forward move of the rth kind for λ_{j_2} is replacing r-marked λ_{j_2} with an r-marked $\lambda_{j_2} + 1$, and hence enabling (d2) for λ_{j_1}. Observe that there are no parts equal to $\lambda_{j_1} + 1 = \lambda_{j_2} - 1$ at all in this case. □

As an example, a backward move of the 1st kind on 1-marked 9 in λ enables a backward move of the 1st kind for 1-marked 11. Also, the hypotheses of case (ii) hold for the 3-marked 3 and 3-marked 6 in λ, but a forward move of the 3rd kind is possible on 3-marked 3 regardless of whether a forward move is performed on 3-marked 6 or not. Therefore the result cannot claim necessity. This observation will be used below.

We provide another example and a non-example for Proposition 2.8(ii). Let

$$\eta = \left\{\begin{matrix} & 2 & & & \\ & 2 & & & \\ 1 & & 3 & & 5 \\ 1 & & 3 & & 5 \\ 1 & & 3 & & 5 \end{matrix}\right\}, \quad \text{and} \quad \eta' = \left\{\begin{matrix} & & & & 5 & & & \\ & & & 4 & & 6 & & 8 \\ & & 3 & & 5 & & 7 & \\ & 2 & & 4 & & 6 & & 8 \\ 1 & & 3 & & 5 & & 7 & \end{matrix}\right\}.$$

For η, let η_{j_1} be the 3-marked 3, and η_{j_2} be the 3-marked 5, and for η', let η'_{j_1} be the 4-marked 6, and η'_{j_2} be the 4-marked 8. A forward move of the 3rd kind on η_{j_2} enables a forward move of the 3rd kind on η_{j_1}, in spite of the fact that there are 4- and 5-marked 2s. However, a forward move of the 4th kind on η'_{j_2} does not enable a forward move of the 4th kind on η'_{j_1}, since any such move will alter the invariants (some n_i) of Gordon marking. These last examples indicate that we do need a longer hypothesis for the second part in Proposition 2.8.

Proposition 2.9. (i) *When conditions exist, a single forward or backward move of the rth kind switches the parity of number of occurrences of two consecutive parts.*

(ii) *When conditions exist, two successive backward or forward moves on the same part (hereafter a* double *backward or forward move) either preserve the parities of number of occurrences of each part, or else swap the parities of number of occurrences of a and $a + 2$, which have opposite parities, for some positive integer a.*

Proof. (i) This is obvious, since a part λ_{j_0} is replaced by $\lambda_{j_0} \pm 1$.

(ii) We will prove this for backward moves only, the proof will be complete by Proposition 2.7. There are two possibilities. Either there are $r_0 < r_1 \leq r$ such that r_0-marked λ_{j_0} will be replaced by r_0-marked $\lambda_{j_0} - 1$, and r_0-marked $\lambda_{j_1} = \lambda_{j_0}$ will be replaced by r_1-marked $\lambda_{j_1} - 1 = \lambda_{j_0} - 1$. This gives us the first option.

Otherwise if an r-marked λ_j is first replaced by r-marked $\lambda_j - 1$, and then an $r_0 \leq r$-marked part equal to $\lambda_j - 1$ is replaced by an r_0-marked part equal to $\lambda_j - 2$; by (b) this means for any $r_0 < r$ for which there is an r_0 marked part equal to λ_j, there is an r_0-marked part equal to $\lambda_j - 2$. Moreover, there is no r-marked part equal to $\lambda_j - 2$ while λ_j itself is r-marked. Thus the number of occurrences of λ_j is exactly one more than the number of occurrences of $\lambda_j - 2$, i.e. they have opposite parities. Those parities are swapped by the two successive moves described. With $a = \lambda_j - 2$, this gives us the second option, and proves the assertion. □

For example,

$$\lambda = \left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\},$$

$\downarrow$ after a double backward move of the 3rd kind on 3-marked 6 becomes

$$\lambda^{iv} = \left\{\begin{array}{ccccccccccccc} & 3 & & \mathbf{5} & & & & & & & 16 & & \\ 2 & & \mathbf{4} & & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

The double backward move here swaps the opposite parities of number of occurrences of 4 and 6.

3. Main results

Theorem 3.1 (*Andrews [2, Eq. (2.5)]*). *Let $k \geq 2$, $k \geq a \geq 1$. Then,*

$$\sum_{m,n \geq 0} b_{k,a}(m,n) x^m q^n = \sum_{n_1,\ldots,n_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}} x^{N_1 + \cdots + N_{k-1}}}{(q)_{n_1} \cdots (q)_{n_{k-1}}}, \tag{3.1}$$

where $N_r = n_r + \cdots + n_{k-1}$.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n enumerated by $b_{k,a}(m,n)$. We will find integers $N_1 \geq \cdots \geq N_{k-1}$ such that $m = N_1 + \cdots + N_{k-1}$; a *base partition* $\widetilde{\lambda}$ which is enumerated by $b_{k,a}(m, \widetilde{n})$ where

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1},$$

and for which no further backward moves are possible without violating the conditions for $b_{k,a}(m,n)$; and $k-1$ partitions $\mu^{(r)}$ such that $\mu^{(r)}$ has at most $n_r = N_r - N_{r+1}$ parts, $r = 1, \ldots, k-1$; all unique. This will give us an injective mapping from the partitions enumerated on the left-hand side of identity (3.1) to the partitions enumerated on the right-hand side, where the exponent of q in the numerator will account for $|\widetilde{\lambda}|$, the factors on the denominator will account for $\mu^{(r)}$, $r = 1, \ldots, k-1$, and the exponent of x for the number of parts.

Conversely, let $N_1 \geq \cdots \geq N_{k-1}$, $n_r = N_r - N_{r+1}$, $r = 1, \ldots, k-1$; $k-1$ partitions $\mu^{(r)}$ such that $\mu^{(r)}$ has at most $n_r (= N_r - N_{r+1})$ parts, $r = 1, \ldots, k-1$ be given. Let $m = N_1 + \cdots + N_{k-1}$. We will first construct a *base partition* $\widetilde{\lambda}$ which is enumerated by $b_{k,a}(m, \widetilde{n})$ where

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1},$$

and for which no further backward moves are possible without violating the conditions for $b_{k,a}(m,n)$. Then we will produce a partition λ of

$$n = |\lambda| = N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1} + \sum_{r=1}^{k-1} |\mu^{(r)}|$$

enumerated by $b_{k,a}(m, n)$, again uniquely. This in turn will give us an injective mapping from the partitions listed on the right-hand side of (3.1) to the partitions listed on the left hand side.

Then we will argue that the two constructions are inverse to each other, proving the theorem bijectively.

Given $\lambda = (\lambda_1, \ldots, \lambda_m)$, a partition of n enumerated by $b_{k,a}(m, n)$, let N_r be the number of r-marked parts in the Gordon marking of λ. Let $n_r = N_r - N_{r+1}$, $r = 1, \ldots, k-1$. Observe that $0 = N_k = N_{k+1} = \ldots$, since there are no more than $k-1$ occurrences of any two consecutive parts. Let $\lambda_{j_{n_{k-1}}} < \cdots < \lambda_{j_1}$ be the $k-1$ marked parts. Then a backward move is possible on $\lambda_{j_{n_{k-1}}}$ unless $\lambda_{j_{n_{k-1}}} = 1$. This cannot happen when $a < k$, by the hypothesis. We make a sequence of backward moves of the $(k-1)$th kind on $\lambda_{j_{n_{k-1}}}$, until the partition ends with

$$\left\{\begin{array}{cccc} & 2 & \uparrow & \\ & \vdots & (k-a) & \\ & 2 & \downarrow & \\ 1 & & \uparrow & \cdots \\ \vdots & & (a-1) & \\ 1 & & \downarrow & \end{array}\right\}.$$

Note that with this configuration, one more backward move of the $(k-1)$th kind on $(k-1)$-marked 2 will bring exactly a 1's in the partition, violating a condition for $b_{k,a}(m, n)$. We call the number of required moves $\mu^{(k-1)}_{n_{k-1}}$. By Proposition 2.8 and the following remark, we can perform at least $\mu^{(k-1)}_{n_{k-1}}$ backward moves on $\lambda_{j_{n_{k-1}-1}}$, possibly more. So we call the number of backward moves of the $(k-1)$th kind on $\lambda_{j_{n_{k-1}-1}}$ required to make the partition end with

$$\left\{\begin{array}{ccccc} & 2 & & 4 & \uparrow & \\ & \vdots & & \vdots & (k-a) & \\ & 2 & & 4 & \downarrow & \\ 1 & & 3 & & \uparrow & \cdots \\ \vdots & & \vdots & & (a-1) & \\ 1 & & 3 & & \downarrow & \end{array}\right\}$$

$\mu^{(k-1)}_{n_{k-1}-1}$. No more backward moves of the $(k-1)$th kind on $(k-1)$-marked 4 are possible, since (b) is not satisfied.

We repeat this process for the remaining $(k-1)$-marked parts in their increasing order, forming the partition $\mu^{(k-1)}$ with at most n_{k-1} parts $\mu^{(k-1)}_{n_{k-1}} \leq \cdots \leq \mu^{(k-1)}_1$. So far the transformed λ looks like

$$\left\{\begin{array}{ccccccc} & \multicolumn{5}{c}{\overbrace{\hspace{12em}}^{(k-1)n_{k-1}\text{ parts}}} & \\ & 2 & & 4 & & 2N_{k-1} & \uparrow & \\ & \vdots & & \vdots & & \vdots & (k-a) & \\ & 2 & & 4 & & 2N_{k-1} & \downarrow & \text{parts} \\ 1 & & 3 & & \cdots (2N_{k-1}-1) & & \uparrow & \geq (2N_{k-1}+1) \\ \vdots & & \vdots & & \vdots & & (a-1) & \\ 1 & & 3 & & (2N_{k-1}-1) & & \downarrow & \end{array}\right\},$$

and we have made a total of $|\mu^{(k-1)}|$ backward moves of the $(k-1)$th kind.

We work on an example on the fly.

Example. We take λ as given far above, noting that it is a partition listed by $b_{4,3}(17, 162)$. Namely, $k-1 = 3$, $a-1 = 2$, $m = 17$ and $n = 162$. By the Gordon marking of λ, $N_1 = 8$, $N_2 = 6$, and $N_3 = 3$. Thus, $n_1 = 2$, $n_2 = 3$, and $n_3 = 3$. Following the above notation, $\lambda_{j_3} =$ 3-marked 3, $\lambda_{j_2} =$ 3-marked 6, and $\lambda_{j_1} =$ 3-marked 16. It is easy to see that after $\mu_3^{(3)} = 3$ backward moves of the 3rd kind on 3-marked 3, and $\mu_2^{(3)} = 5$ moves of the 3rd kind on 3-marked 6, λ will be made into

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & & & 16 & & \\ 1 & & 3 & & & 7 & & & 13 & 15 & & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

Itwould be instructive to make the backward moves of the 3rd kind on 3-marked 16 one by one. The boldfaced part at each step is the decreased one.

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & & & & & \mathbf{15} & & \\ 1 & & 3 & & & 7 & & & 13 & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & & \mathbf{14} & & & \\ 1 & & 3 & & & 7 & & & 13 & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & & & & \mathbf{13} & & & \\ 1 & & 3 & & & 7 & & & 13 & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & & 13 & & & \\ 1 & & 3 & & & 7 & & & \mathbf{12} & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & \mathbf{12} & & & & \\ 1 & & 3 & & & 7 & & & 12 & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & & 13 & 15 & 17 & \end{array}\right\}$$

$\downarrow$ After 10 more analogous moves, we obtain

$$\left\{\begin{array}{ccccccccccc} & 2 & & 4 & \mathbf{6} & & & & & & \\ 1 & & 3 & & 6 & 9 & & & 15 & & 18 \\ 1 & & 3 & & 6 & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}.$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & 6 & & & & & & \\ 1 & & 3 & & & 6 & 9 & & & 15 & & 18 \\ 1 & & 3 & & \mathbf{5} & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & 6 & & & & & & \\ 1 & & 3 & & \mathbf{5} & & 9 & & & 15 & & 18 \\ 1 & & 3 & & 5 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

We have made $\mu_1^{(3)} = 17$ backward moves of the 3rd kind on 3-marked 16, and have reached a configuration in which no more backward moves of the 3rd kind are possible without violating the conditions for $b_{4,3}(m, n)$. We have made a total of $25 = |\mu^{(3)}|$ backward moves of the 3rd kind.

We repeat the above process for $r = k - 2, \ldots, 1$ in decreasing order as follows: For each r, we choose the n_r largest r-marked parts $\lambda_{j_{n_r}} < \cdots < \lambda_{j_1}$ beginning with the smallest of these, and pick the next smallest after we are done with the one at hand. We perform $\mu_i^{(r)}$ backward moves of the rth kind, $i = n_r, \ldots, 1$, so that no more moves are possible without violating the conditions for $b_{k,a}(m, n)$.

When all $\mu^{(r)}$ are constructed, λ will be transformed into

$$\widetilde{\lambda} = \left\{ \overbrace{\begin{array}{lccc} & 2 & & 2N_{k-1} \\ & \vdots & & \vdots \\ & 2 & & 2N_{k-1} \\ 1 & \cdots & (2N_{k-1}-1) & \\ \vdots & & \vdots & \\ 1 & & (2N_{k-1}-1) & \end{array}}^{(k-1)n_{k-1}\text{ parts}} \begin{array}{c} \uparrow \\ (k-a) \\ \downarrow \\ \uparrow \\ (a-1) \\ \downarrow \end{array} \cdots \right.$$

$$\cdots \overbrace{\begin{array}{cccc} & 2N_r & & (2N_{r+1}+2) \\ & \vdots & & \vdots \\ & 2N_r & & (2N_{r+1}+2) \\ (2N_r-1) & & \cdots\ (2N_{r+1}+1) & \\ \vdots & & \vdots & \\ (2N_r-1) & & (2N_{r+1}+1) & \end{array}}^{rn_r\text{ parts}} \begin{array}{c} \uparrow \\ \max\{r-(a-1),0\} \\ \downarrow \\ \uparrow \\ \min\{r,a-1\} \\ \downarrow \end{array} \cdots$$

$$\left. \cdots \overbrace{\begin{array}{c} \\ \\ \\ (2N_2+1), (2N_2+3), \ldots, (2N_1-1) \end{array}}^{n_1\text{ parts}} \right\}.$$

It is straightforward to check that (b) is not satisfied for any part in $\widetilde{\lambda}$ for a backward move of some kind, save for the $(k-1)$-marked 2. Yet, the impossibility of the backward move of the $(k-1)$th kind on $(k-1)$-marked 2 was discussed above. Also, it is readily seen that $\widetilde{\lambda}$ is enumerated by $b_{k,a}(m, \widetilde{n})$. Therefore, $\widetilde{\lambda}$ is the *base partition* we are looking for. It remains to show that

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}.$$

This follows by the Gordon marking of $\widetilde{\lambda}$, for

$$\widetilde{\lambda}^{(r)} = \begin{cases} 1, 3, \ldots, 2N_r - 1 & \text{if } \ r < a \\ 2, 4, \ldots, 2N_r & \text{if } \ r \geq a. \end{cases}$$

Continuing with the above example,

Example. For $r = 2$, the $n_2 = 3$ largest 2-marked parts are $\lambda_{j_3} =$ 2-marked 9, $\lambda_{j_2} =$ 2-marked 15, and $\lambda_{j_1} =$ 2-marked 18. After $\mu_3^{(2)} = 4$ backward moves of the 2nd kind on 2-marked 9, the so far transformed λ looks like

$$\left\{ \begin{array}{cccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & \\ 1 & & 3 & & 5 & & 7 & & & & 15 & & & 18 \\ 1 & & 3 & & 5 & & 7 & & 11 & 13 & 15 & & 17 & \end{array} \right\}.$$

After $\mu_2^{(2)} = 8$ backward moves of the 2nd kind on 2-marked 15, it looks like

$$\left\{\begin{array}{lllllllllllll} & 2 & & 4 & & 6 & & & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & & & & & 18 \\ 1 & & 3 & & 5 & & 7 & 9 & & 13 & 15 & 17 & \end{array}\right\}.$$

And, after $\mu_1^{(2)} = 9$ backward moves of the 2nd kind on 2-marked 18, it looks like

$$\left\{\begin{array}{llllllllllll} & 2 & & 4 & & 6 & & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & 15 & 17 \end{array}\right\}.$$

Finally, for $r = 1$, the largest $n_1 = 2$ of the 1-marked symbols are $\lambda_{j_2} =$ 1-marked 15, and $\lambda_{j_1} =$ 1-marked 17. After the obvious $\mu_2^{(1)} = 2$ and $\mu_1^{(1)} = 2$ backward moves of the 1st kind, which happen to be mere subtractions, we obtain

$$\widetilde{\lambda} = \left\{\begin{array}{llllllllll} & 2 & & 4 & & 6 & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & 13 & 15 \end{array}\right\}.$$

At this point,

$$|\widetilde{\lambda}| = 8^2 + 6^2 + 3^2 + 3 = 112,$$

and

$$\begin{aligned} n = |\lambda| = 162 &= N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1} + \sum_{r=1}^{k-1} |\mu^{(r)}| \\ &= 112 + |\mu^{(3)}| + |\mu^{(2)}| + |\mu^{(1)}| \\ &= 112 + (17 + 5 + 3) + (9 + 8 + 4) + (2 + 2), \end{aligned}$$

as desired.

Conversely, let $n_r \geq 0$, partitions $\mu^{(r)}$ with at most n_r parts, $r = 1, \ldots, k-1$ be given. Let $N_r = n_r + \cdots + n_{k-1}$, and let $\widetilde{\lambda}$ be a base partition such that

$$\widetilde{\lambda}^{(r)} = \begin{cases} 1, 3, \ldots, 2N_r - 1 & \text{if } \ r < a \\ 2, 4, \ldots, 2N_r & \text{if } \ r \geq a. \end{cases}$$

Then, as above,

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}.$$

$\widetilde{\lambda}$ has $m = N_1 + \cdots + N_{k-1}$ parts. It is enumerated by $b_{k,a}(m, \widetilde{n})$, and any backward move of any kind, if at all possible, would violate the conditions for $b_{k,a}(m, \widetilde{n})$.

We take $\mu^{(1)} = \mu_1^{(1)} \geq \cdots \geq \mu_{n_1}^{(1)}$, and the largest n_1 of the 1-marked parts. (c) and (d2) (possibly (d1)) are always satisfied for the largest 1-marked part. Also, for the remaining largest $n_1 - 1$ 1-marked parts, the hypotheses of Proposition 2.8(ii) are satisfied. Backed by Proposition 2.8, we perform $\mu_i^{(1)}$ forward moves of the 1st kind on the ith largest 1-marked part for $j = 1, \ldots, n_1$ beginning with the largest and then picking the next largest once we are done with the one at hand.

Then, for $r = 2, \ldots, k-1$ in increasing order, we observe that (c) and (d2) are always satisfied for the largest of the r-marked parts provided that $n_r \geq 1$, and the hypotheses of Proposition 2.8(ii) hold for the next largest $n_r - 1r$-marked parts when $n_r > 1$. So we take $\mu^{(r)} = \mu_1^{(r)} \geq \cdots \geq \mu_{n_r}^{(r)}$ and perform $\mu_i^{(r)}$ forward moves of the rth kind on the ith largest r-marked part for $j = 1, \ldots, n_r$ beginning with the largest and then picking the next largest once we are done with the one at hand. We call the final partition λ, after all moves are made.

By virtue of forward moves, $N_1, \ldots, N_{k-1}$ remain invariant, and hence the conditions for $b_{k,a}(m, n)$ are satisfied for λ, and by construction,

$$n = |\lambda| = N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1} + \sum_{r=1}^{k-1} |\mu^{(r)}|,$$

as claimed.

Finally, we note that in the two constructions above the steps are performed in the exact reverse order. By the observation following Proposition 2.7, the two transformations are inverse to each other. □

The example given for the first part of the proof can be worked backwards.

The following result is a refinement to Theorem 3.1. Andrews established it analytically in [7, Eqs. (3.3), (4.2), (4.3)] for k and a having the same parity. In this sense, this version is a slight generalization.

Theorem 3.2 (*Andrews*). *Let $k \geq 2$, $k \geq a \geq 1$. Then,*

$$\sum_{m,n \geq 0} w_{k,a}(m, n) x^m q^n = \sum_{n_1,\ldots,n_{k-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+2N_a+2N_{a+2}+\cdots} x^{N_1+\cdots+N_{k-1}}}{(q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{k-1}}}. \tag{3.2}$$

Proof. Observe that by Proposition 2.9(ii), if in both λ and λ' the number of occurrences of even numbers are all even (or just as well all odd), and if $\lambda \rightsquigarrow \lambda'$ by a sequence of forward or backward moves of any kind, then the number of moves made must be even.

Imitating the proof of Theorem 3.1, we begin with a given λ enumerated by $w_{k,a}(m, n)$. we perform double backward moves of some kind at each step, instead of single ones. This will ensure that we obtain $\mu^{(r)}$ with all even parts, $r = 1, \ldots, k-1$. Now, if $r \geq a$, and $r - (a-1)$ is odd, we miss $\widetilde{\lambda}$ by exactly n_r single moves to be performed on the n_r largest r-marked parts by the remark at the beginning of the proof. In other words, for $r = a, a+2, \ldots$, the largest $n_r r$-marked parts are even numbers that occur an odd number of times. This will account for $n_a + n_{a+2} + \cdots$ backward moves on the total. Therefore, $\mu^{(r)}$ must be accompanied by a separate n_r for $r = a, a+2, \ldots$.

Conversely, beginning with $\widetilde{\lambda}$ as in the proof of Theorem 3.1, for the moves on the n_r largest r-marked parts $r = a, a+2, \ldots$, we first need to perform n_r single moves on each, so as to ensure that all even parts occur an even number of times. Then we continue with making half as many double moves as parts of $\mu^{(r)}$, $r = 1, \ldots, k-1$. Again, by the remark at the beginning of the proof, the constructed λ will satisfy the conditions for $w_{k,a}(m, n)$.

The fact that the two constructions above are inverse to each other follows by the argument employed in the last part of the proof of Theorem 3.1.

The contribution of the extra moves should be added to the exponent of q on the numerator on the right-hand side:

$$\begin{aligned} &N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1} + n_a + n_{a+2} + \cdots \\ &\quad = N_1^2 + \cdots + N_{k-1}^2 + 2N_a + 2N_{a+2} + \cdots. \end{aligned}$$

And the result follows. □

We give a simpler example here, and work backwards.

Example. Let $k - 1 = 3$, $a - 1 = 1$, $N_1 = 4$, $N_2 = 3$, and $N_1 = 1$. Let $\mu^{(1)} = 2$, $\mu^{(2)} = 4, 2$, and $\mu^{(3)} = 8$. Then the base partition described as in the proof of Theorem 3.1 will be

$$\widetilde{\lambda} = \left\{ \begin{matrix} & 2 & & & & & \\ & 2 & & 4 & & 6 & \\ 1 & & 3 & & 5 & & 7 \end{matrix} \right\}.$$

We begin with $\mu^{(1)}$, and simply make a $2/2 = 1$ double move on the largest 1-marked part 7.

$$\left\{ \begin{matrix} & 2 & & & & & & & \\ & 2 & & 4 & & 6 & & & \\ 1 & & 3 & & 5 & & & & 9 \end{matrix} \right\}.$$

For $r = 2$, $r - (a - 1) = 2 - 1 = 1$ is odd. Thus we need to make $n_2 = 2$ single forward moves of the 2nd kind on the 2 largest 2-marked parts 4 and 6.

$$\left\{\begin{array}{ccccc} & 2 & & & \\ & 2 & 4 & 6 & \\ 1 & & 4 & 6 & 9 \end{array}\right\}.$$

Then we can realize $\mu^{(2)}$ as $4/2 = 2$, and $2/2 = 1$ double forward moves of the 2nd kind on 2-marked 6 and 2-marked 4, respectively. 2-marked 6 goes first, as mentioned in the above proofs, and here we see why: both (d1) and (d2) fail for 2-marked 4.

$$\left\{\begin{array}{ccccc} & 2 & & & \\ & 2 & 5 & & 9 \\ 1 & & 5 & 7 & 9 \end{array}\right\}.$$

Finally, we perform $8/2 = 4$ double moves on 3-marked 2.

$$\left\{\begin{array}{ccccc} & & 6 & & \\ 3 & & 6 & & 9 \\ 3 & 5 & & 7 & 9 \end{array}\right\},$$

which is a partition enumerated by $w_{4,2}(8, 48)$.

We also see that the order the moves are made is very important. If we introduce the $n_2 = 2$ single forward moves of the 2nd kind in advance of introducing any $\mu^{(r)}$, such as

$$\left\{\begin{array}{ccccc} & 2 & & & \\ & 2 & & 5 & 7 \\ 1 & & 3 & 5 & 7 \end{array}\right\},$$

then no forward moves of the 1st kind are possible anymore, and $\mu^{(1)}$ is rendered useless. Although a specific order of forward moves may exist so that all $\mu^{(r)}$ can be realized as forward moves, this violates the uniqueness of the constructions above. Furthermore, this specific order does not exist in general.

The following result is a slight generalization of the identity that follows from [7, Eqs. (3.4), (4.5), (4.27)], where it is shown for k odd and a even.

Theorem 3.3 (*Andrews*). *Let $k \geq a \geq 2$, and a be even. Then,*

$$\sum_{m,n\geq 0} \overline{w}_{k,a}(m,n)x^m q^n = \sum_{m,n\geq 0} \overline{w}_{k,a-1}(m,n)x^m q^n \tag{3.3}$$

$$= \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{a-1}+\cdots N_{k-1}+n_1+n_3+\cdots+n_{a-3}} x^{N_1+\cdots+N_{k-1}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}}. \tag{3.4}$$

Proof. (3.3) is obvious, since 1 cannot appear exactly $(a - 1)$ (an odd number of) times. For (3.4), we argue as in the proof of Theorem 3.2. The only difference is that the extra single moves are required for the n_r largest r-marked odd parts, $r = 1, 3, \ldots, a-3$. Because in the base partition $\widetilde{\lambda}$ described in the proof of Theorem 3.1, these are precisely the odd parts that appear an odd number of times, hence the extra $n_1 + n_3 + \cdots + n_{a-3}$ in the exponent of q in the numerator on the rightmost infinite sum. The rest of proof of Theorem 3.2 applies word for word. □

Example. For $k = 6$ and $a - 1 = 5$, given

$n_1 = 2, \quad n_2 = 3, \quad n_3 = 2, \quad n_4 = 0, \quad n_5 = 2,$
$\mu^{(1)} = 2, \quad \mu^{(2)} = 6, 4, 2, \quad \mu^{(3)} = 6, 6,$
$\mu^{(4)} =$ the empty partition, $\quad \mu^{(5)} = 28, 24,$

we will construct a partition λ enumerated by $b_{6,5}(24, 238)$ in which odd parts appear an even number of times. Or in short, a partition enumerated by $\overline{w}_{6,5}(24, 238)$

$$\begin{aligned} &238 = N_1^2 + \cdots + N_5^2 + N_5 + n_1 + n_3 + |\mu^{(1)}| + \cdots + |\mu^{(5)}|, \quad \text{and} \\ &24 = N_1 + \cdots + N_5. \end{aligned}$$

We begin by constructing the base partition defined in the proof of Theorem 3.1.

$$\widetilde{\lambda} = \left\{ \begin{array}{lllllllllll} & 2 & & 4 & & & & & & & \\ 1 & & 3 & & & & & & & & \\ 1 & & 3 & & 5 & 7 & & & & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & 15 & 17 \end{array} \right\}$$

We observe that 5, 7, 15, and 17 are odd parts that appear an odd number of times. We need single forward moves performed on the 1-marked 17, 1-marked 15, 3-marked 7, and 3-marked 5. If the forward moves of the appropriate kind are performed in decreasing order of the mentioned parts, hypotheses of Proposition 2.8(ii) will be satisfied so that the moves are possible. This is the point where the extra n_1 is realized as part of the exponent of q in the numerator in the generating function (3.4). Remember that we do not touch the larger marked parts until all the smaller marked parts are taken care of. The base partition is transformed into

$$\left\{ \begin{array}{lllllllllll} & 2 & & 4 & & & & & & & \\ 1 & & 3 & & & & & & & & \\ 1 & & 3 & & 5 & 7 & & & & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & 16 & 18 \end{array} \right\}.$$

Then we take into account the partitions $\mu^{(r)}$ as described in the proof of Theorem 3.1. After realizing $\mu^{(1)}$, we have

$$\left\{ \begin{array}{lllllllllll} & 2 & & 4 & & & & & & & \\ 1 & & 3 & & & & & & & & \\ 1 & & 3 & & 5 & 7 & & & & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & 16 & 20 \end{array} \right\}.$$

$\downarrow$ After applying $\mu^{(2)}$, we have

$$\left\{ \begin{array}{lllllllllll} & 2 & & 4 & & & & & & & \\ 1 & & 3 & & & & & & & & \\ 1 & & 3 & & 5 & 7 & & & & & \\ 1 & & 3 & & 5 & 7 & 10 & & 14 & 17 & \\ 1 & & 3 & & 5 & 7 & 10 & 12 & 14 & 17 & 20 \end{array} \right\}.$$

At this point, we realize the extra n_3 as part of the exponent of q in the numerator. We perform single forward moves of the 3rd kind on the largest 2 of the 3-marked parts. There are no more odd parts that appear an odd number of times.

$$\left\{ \begin{array}{lllllllllllll} & 2 & & 4 & & & & & & & & & \\ 1 & & 3 & & & & & & & & & & \\ 1 & & 3 & & & 6 & & 8 & & & & & \\ 1 & & 3 & & 5 & & 7 & & 10 & & 14 & 17 & \\ 1 & & 3 & & 5 & & 7 & & 10 & 12 & 14 & 17 & 20 \end{array} \right\}$$

$\downarrow$ And after applying $\mu^{(3)}$, we have

$$\left\{ \begin{array}{lllllllllllll} & 2 & & 4 & & & & & & & & & \\ 1 & & 3 & & & & & & & & & & \\ 1 & & 3 & & & & 9 & & 11 & & & & \\ 1 & & 3 & & 6 & & 9 & & 11 & & 14 & 17 & \\ 1 & & 3 & & 6 & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}.$$

We keep in mind that we interpret the parts of $\mu^{(r)}$ as half as many double moves instead of as many single moves unlike in the example in the proof of Theorem 3.1. It is instructive to go step by step for performing 14 double forward moves of 5th kind (28 moves) on the 5-marked 4.

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & 4 \\ 1 & & & 4 \\ 1 & & & 4 & & & & & 9 & & 11 \\ 1 & & 3 & & & 6 & & & 9 & & 11 & & 14 & 17 \\ 1 & & 3 & & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & 4 \\ 1 & & & 4 \\ 1 & & & 4 & & & & & 9 & & 11 \\ 1 & & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & & 5 \\ 1 & & & & 5 \\ 1 & & & 4 & & & & & 9 & & 11 \\ 1 & & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & & & 6 \\ 1 & & & & 5 \\ 1 & & & & 5 & & & & 9 & & 11 \\ 1 & & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & & & 6 \\ 1 & & & & & 6 \\ 1 & & & & & 6 & & & 9 & & 11 \\ 1 & & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & & & & 7 \\ 1 & & & & & & 7 \\ 1 & & & & & 6 & & & 9 & & 11 \\ 1 & & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & & & & 7 \\ 1 & & & & & & 7 \\ 1 & & & & & & 7 & & 9 & & 11 \\ 1 & & & 4 & & & 7 & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}$$

$\downarrow$ After 5 more double forward moves of the 5th kind, we obtain

$$\left\{ \begin{array}{ccccccccccccccc} & 2 & & & & & & & & & & 12 \\ 1 & & & & & & & & & & & 12 \\ 1 & & & & & & 7 & & 9 & & 11 \\ 1 & & & 4 & & & 7 & & 9 & & 11 & & 14 & 17 \\ 1 & & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20 \end{array} \right\}.$$

$$\downarrow$$

$$\left\{\begin{array}{llllllllllll}
 & 2 & & & & & & & 12 & & & \\
1 & & & & & & & & 12 & & & \\
1 & & & & 7 & & 9 & & 12 & & & \\
1 & & 4 & & 7 & & 9 & & 12 & 14 & 17 & \\
1 & & 4 & 6 & & 8 & & 10 & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{lllllllllllll}
 & 2 & & & & & & & & 13 & & & \\
1 & & & & & & & & & 13 & & & \\
1 & & & & 7 & & 9 & & 12 & & & & \\
1 & & 4 & & 7 & & 9 & & 12 & & 14 & 17 & \\
1 & & 4 & 6 & & 8 & & 10 & 12 & & 14 & 17 & 20
\end{array}\right\}.$$

Finally, we make 12 double forward moves of the 5th kind (24 moves) on the 5-marked 2 to get

$$\lambda = \left\{\begin{array}{lllllllllllll}
 & & & & & & & 10 & & 13 & & & \\
 & & & & & & & 10 & & 13 & & & \\
 & & 5 & & 7 & & 9 & & 12 & & & & \\
2 & & 5 & & 7 & & 9 & & 12 & & 14 & 17 & \\
2 & 4 & & 6 & & 8 & & 10 & 12 & & 14 & 17 & 20
\end{array}\right\},$$

indeed a partition enumerated by $b_{6,5}(24, 238)$ in which odd parts appear an even number of times. That is, it is a partition enumerated by $\overline{w}_{6,5}(24, 238)$.

Before the next important theorem, we need an auxiliary result.

Lemma 3.4. *Lists of exactly l elements $0 \leq r_1 < \cdots < r_l < n$ weighted by the sum of the smallest numbers in each maximal sublist of consecutive integers are generated by the Gaussian Polynomial* $\begin{bmatrix} n \\ l \end{bmatrix} = \frac{(q)_n}{(q)_l (q)_{n-l}}$.

By a maximal sublist of consecutive integers, we mean a sublist that is not properly contained in any such other that is strictly larger. For instance, let the list $\{1, 2, 4, 5, 6, 9\}$ where $n = 11$ and $l = 6$ be given. The maximal sublists of consecutive integers are $\{1, 2\}$, $\{4, 5, 6\}$, and $\{9\}$, but not, say $\{5, 6\}$. This list is weighted by $1 + 4 + 9 = 14$.

Proof. We will be using the fact that $\begin{bmatrix} n \\ l \end{bmatrix}$ generates partitions into at most l parts, all $\leq n - l$ [5, Thm. 3.1].

Given $0 \leq r_1 < \cdots < r_l < n$, we will construct a unique partition enumerated by $\begin{bmatrix} n \\ l \end{bmatrix}$ of the weight of the given list. Conversely, given a partition enumerated by $\begin{bmatrix} n \\ l \end{bmatrix}$, we will uniquely construct a list $0 \leq r_1 < \cdots < r_l < n$ the weight of which is the number being partitioned. Then we will argue that the transformations are inverse to each other.

Given $0 \leq r_1 < \cdots < r_l < n$, let $\lambda_1 = r_1$. For $i = 2, \ldots, l$ in their increasing order; if $r_i = r_{i-1} + 1$ (i.e. r_i is not the smallest element in a maximal sublist of consecutive integers), then $\lambda_i = 0$ (since r_i does not contribute to the weight). Otherwise if $r_i > r_{i-1} + 1$ (so that r_i is the smallest element in a maximal sublist of consecutive integers), then

$$\begin{aligned}
\lambda'_i &= \lambda_{i-1} + 1, \\
\lambda'_{i-1} &= \lambda_{i-2} + 1, \\
&\vdots \\
\lambda'_2 &= \lambda_1 + 1, \\
\lambda'_1 &= r_i - (i - 1).
\end{aligned}$$

That $\lambda'_i \leq \cdots \leq \lambda'_2$ is immediate. In addition, we need to show that $\lambda'_2 \leq \lambda'_1$, i.e. $\lambda_1 \leq r_i - i$. By construction $\lambda_1 = r_{i_0} - (i_0 - 1)$ for some $i_0 < i$, and by the hypothesis r_i is not in the consecutive list of integers beginning with r_{i_0}. So $r_{i_0} + (i - i_0) < r_i$, which implies $r_{i_0} - (i_0 - 1) \leq r_i - i$, and hence

$\lambda_2' \leq \lambda_1'$. The total contribution is r_i if r_i is the smallest element in a maximal sublist of consecutive integers, zero otherwise. Then we call $\lambda_i'\lambda_i$ and start over for $i+1$ unless $i = l$. At the end, we have a partition $\lambda_l \leq \cdots \leq \lambda_1$ possibly containing zeros, and the sum of all parts equals the weight of the given list. Also, $\lambda_1 = r_l - (l-1)$ together with the fact that $r_l < n$ implies that $\lambda_1 \leq n - l$. Therefore the constructed partition is one enumerated by $\begin{bmatrix} n \\ l \end{bmatrix}$.

Conversely, given $0 \leq \lambda_l \leq \cdots \leq \lambda_1 \leq n - l$, let $j = l$. Let i be such that $\lambda_i \neq 0$ but $\lambda_{i+1} = \cdots = \lambda_j = 0$. Set

$$\begin{aligned} r_i &= \lambda_1 + (i-1), \\ r_{i+1} &= r_i + 1, \\ r_{i+2} &= r_{i+1} + 1, \\ &\vdots \\ r_j &= r_{j-1} + 1, \end{aligned}$$

and then substitute

$$\begin{aligned} \lambda_1 &\to \lambda_2 - 1, \\ \lambda_2 &\to \lambda_3 - 1, \\ &\vdots \\ \lambda_{i-1} &\to \lambda_i - 1, \\ \lambda_i &\to 0. \end{aligned}$$

Then, set $j = i - 1$ and start over until $i = 1$.

If $i \neq 1$ in the first run, then there is an $i_0 < i$, $r_{i_0} < r_i$ for which in the original partition $\lambda_1 = r_i - (i-1)$ and $\lambda_2 = r_{i_0} - (i_0 - 1) + 1$. $\lambda_2 \leq \lambda_1$ implies $r_{i_0} - i_0 \leq r_i - i - 1$, and hence $r_{i_0} + i - i_0 < r_i$. That is, r_i is not in the sublist of consecutive integers containing r_{i_0} for any $i_0 < i$, but $\{r_i, r_{i+1}, \ldots, r_j\}$ form a maximal sublist of consecutive integers. A repeated application of this argument shows that $\lambda_1 + \cdots + \lambda_l$ would be the weight of the constructed sublist, since for each smallest element r_i in a maximal sublist of consecutive integers, r_i is extracted from the partition.

The above assignments or substitutions together with the order they are performed clearly show that the described operations are inverse to each other. $\square$

Remark. This can be equivalently done using the notion of a *hook*, and adjunction of hooks to partitions under suitable conditions.

Example. We construct a partition enumerated by $\begin{bmatrix} 11 \\ 6 \end{bmatrix}$ using the list $\{1, 2, 4, 5, 6, 9\}$.

To begin, $\lambda_1 = r_1 = 1$.
$2 = 1 + 1$, so $\lambda_2 = 0$.
$4 \neq 2 + 1$, so

$$\lambda_3 = \lambda_2 + 1 = 1,$$
$$\lambda_2 = \lambda_1 + 1 = 2,$$
$$\lambda_1 = 4 - 2 = 2.$$

$5 = 4 + 1$, so $\lambda_4 = 0$.
$6 = 5 + 1$, so $\lambda_5 = 0$.
$9 \neq 6 + 1$, so

$$\lambda_6 = \lambda_5 + 1 = 1,$$
$$\lambda_5 = \lambda_4 + 1 = 1,$$
$$\lambda_4 = \lambda_3 + 1 = 2,$$
$$\lambda_3 = \lambda_2 + 1 = 3,$$
$$\lambda_2 = \lambda_1 + 1 = 3,$$
$$\lambda_1 = 9 - 5 = 4.$$

Thus,

$$\lambda = 1 \leq 1 \leq 2 \leq 3 \leq 3 \leq 4,$$

with 6 (≤ 6) parts all $\leq 11 - 6 = 5$, and

$$|\lambda| = 1 + 1 + 2 + 3 + 3 + 4 = 14 = 1 + 4 + 9 = 14$$

is the weight of the given list, as claimed.

Theorem 3.5 (*Andrews' Conjecture*). *Given $1 \leq a \leq k$, $2 \leq k$,*

$$\sum_{n_1, \ldots, n_{k-1} \geq 0} q^{N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}} \quad x^{N_1 + \cdots + N_{k-1}} \quad y^{n_a + n_{a+2} + \cdots} \quad \cdots$$
$$\times \frac{H_{n_1} \cdots H_{n_{a-1}} \widetilde{H}_{n_a} H_{n_{a+1}} \widetilde{H}_{n_{a+2}} \cdots}{(q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{k-1}}} \tag{3.5}$$

generates the partitions enumerated by $b_{k,a}(m, n)$, where the exponent of x accounts for the number of parts, the exponent of y accounts for the number of even parts that appear an odd number of times, and the H are the Rogers–Szegő polynomials [16]

$$H_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_2 (qy)^j,$$

and

$$\widetilde{H}_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_2 \left(\frac{q}{y}\right)^j.$$

Here

$$\begin{bmatrix} n \\ j \end{bmatrix}_2 = \frac{(q^2; q^2)_n}{(q^2; q^2)_j (q^2; q^2)_{n-j}}$$

generates partitions into at most j even parts, all $\leq 2n - 2j$.

For $k = 2$ and 3, and $a = k$, this is [7, Thm. 4, Eq. (5.1)], and Andrews conjectured that the identity holds for all $a = k \geq 2$ in the same paper.

Proof. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, a partition of n enumerated by $b_{k,a}(m, n)$; for $r = 1, \ldots, k - 1$, we will produce N_r, $n_r = N_r - N_{r+1} \geq 0$;

a base partition $\widetilde{\lambda}$ as in the proof of Theorem 3.1 (the factor $q^{N_1^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_{k-1}} x^{N_1 + \cdots + N_{k-1}}$);

partitions $\mu^{(r)}$ having at most n_r all even parts (the factors $\frac{1}{(q^2;q^2)_{n_r}}$);

partitions $\nu^{(r)}$ having exactly j_r odd parts $\leq 2n_r - 2j_r + 1$, $0 \leq j_r \leq n_r$ accounting for y^{j_r} or $y^{n_r - j_r}$ as appropriate (the factor H_{n_r} or $y^{n_r} \widetilde{H}_{n_r}$).

Conversely, given $n_1, \ldots, n_{k-1} \geq 0$;

partitions $\mu^{(r)}$ having at most n_r all even parts;

partitions $\nu^{(r)}$ having exactly j_r odd parts $\leq 2n_r - 2j_r + 1$, $0 \leq j_r \leq n_r$;

we will first construct a base partition $\widetilde{\lambda}$ as in the proof of Theorem 3.1, and then recover a partition λ enumerated by $b_{k,a}(m, n)$, where there is a factor of y raised to a power equal to the number of even parts that appear an odd number of times.

Finally, we will argue that the two transformations are inverses to each other.

Proposition 2.9 implies that double backward or forward moves keep the number of even parts that appear an odd number of times invariant. This is not true for those that appear an even number of times, since zero is just as even. Otherwise if we wanted to keep track of the even parts that appear a positive even number of times, the following small example shows that the described approach is not enough.

$$\begin{Bmatrix} 2 \\ 2 \end{Bmatrix} \text{ becomes } \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

after a double backward move of the 2nd kind.

For each $r = k-1, k-2, \ldots, 1$ in decreasing order, we consider the n_r largest r-marked parts $\lambda_{l_1} < \cdots < \lambda_{l_{n_r}}$ and perform so many double backward moves of the rth kind on $\lambda_{l_1}, \lambda_{l_2}, \ldots, \lambda_{l_{n_r}}$ in their increasing order so that no more double backward moves of the rth kind is possible as in the proofs of Theorems 3.2 and 3.3. We obtain $\mu^{(r)}$ here. The resulting partition does not have to be the base partition we are looking for, since single backward moves may still be possible which will give rise to more double moves. Then, we look at $\lambda_{l_1}, \lambda_{l_2}, \ldots, \lambda_{l_{n_r}}$ one by one in increasing order. There are two cases.

(i) $r > a-1$ and $r \equiv a-1 \pmod 2$, or $r \leq a-1$

In this case, the base partition $\widetilde{\lambda}$ has its n_r largest r-marked parts even numbers that appear an even number of times ($r > a-1$ and $r \equiv a-1 \pmod 2$), or odd numbers ($r \leq a-1$).

If λ_{l_i} is an even number that appears an even number of times, or is an odd number, we do not do anything, and continue with $\lambda_{l_{i+1}}$ unless $i = n_r$. Because λ_{l_i} is already where it would be in $\widetilde{\lambda}$, the base partition, no further backward moves of the rth kind are possible on it.

Else if λ_{l_i} is an even number that appears an odd number of times, we perform one single move on it. If $\lambda_{l_{i+1}}$ also is an even number that appears an odd number of times, we restart the procedure for $\lambda_{l_{i+1}}$. Otherwise, if $\lambda_{l_{i+1}}$ is an even number that appears a even number of times, or an odd number, then by (b) the single backward move of the rth kind on λ_{l_i} will enable a double backward move of the rth kind on $\lambda_{l_{i+1}}$, hence on $\lambda_{l_{i+2}}, \lambda_{l_{i+3}}, \ldots, \lambda_{l_{n_r}}$ by Proposition 2.8. We first perform these double backward moves, then proceed with $\lambda_{l_{i+2}}$.

The single backward move yields the factor qy, since an even part that appears an odd number of times is accounted for.

(ii) $r > a-1$ and $r \not\equiv a-1 \pmod 2$

In this case, the base partition $\widetilde{\lambda}$ has its n_r largest r-marked parts even numbers that appear $r-(a-1)$ (an odd number of) times. This explains the separate factor y^{n_r}. We proceed exactly as in the preceding case except that we check if λ_{l_i} is an even number that appears an *even* number of times to perform a single backward moves of the rth kind, and if so we check if $\lambda_{l_{i+1}}$ is an even number that appears an *odd* number of times to perform a sequence of double backward moves of the rth kind on r-marked parts strictly greater than λ_{l_i}. Single backward moves of the rth kind bring factors $\frac{q}{y}$. This is because each described single backward move of the rth kind switches the number of occurrences of an even part from even into odd.

Now, if the i_1th, i_2th, $\ldots$, i_{j_r}th largest parts required single backward moves of the rth kind among $\lambda_{l_1}, \lambda_{l_2}, \ldots, \lambda_{l_{n_r}}$, then we have also made $i_s - 1$ double backward moves of the rth kind if $i_{s+1} \neq i_s - 1$. In other words, for the list $\{i_{j_r} - 1, \ldots, i_2 - 1, i_1 - 1\}$, we require $i_s - 1$ double backward moves of the rth kind if $i_s - 1$ is the smallest element in a maximal sublist of consecutive integers in the list. Lemma 3.4 applies to give a corresponding partition listed by $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2$. Subscript 2 is due to double moves instead of single ones. Along with the single moves and the y factors, we have $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (qy)^{j_r}$ (case (i)), or $y^{n_r} \begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (\frac{q}{y})^{j_r}$ (case (ii)). Either generates partitions with j_r odd parts $\leq 2n_r - 2j_r + 1$. This gives us $\nu^{(r)}$ as described above. Summing over all possible j_r yields H_{n_r} (case (i)), or $y^{n_r} \widetilde{H}_{n_r}$ (case (ii)). This gives us an injection from the partitions enumerated by $b_{k,a}(m,n)$ to partitions enumerated by (3.5), and concludes the first half of the construction.

As in the proof of Theorem 3.1, let us work on an example on the fly.

Example. Let

$$\lambda = \left\{ \begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

be a partition enumerated by $b_{4,3}(17, 162)$. In this context, it is accompanied by a factor y^3, since 4, 16 and 18 are precisely the even parts that appear an odd number of times. After making 2 backward moves of the 3rd kind on the $r = 3$-marked 3, 4 moves on 3-marked 6, and 16 moves on 3-marked 16, (1, 2, and 8 double backward moves of the 3rd kind, respectively) we obtain

$$\left\{ \begin{array}{cccccccccccc} & 2 & & 4 & 6 & & & & & & & \\ & 2 & & 4 & 6 & & 9 & & & 15 & & 18 \\ 1 & & 3 & & 6 & 8 & & 11 & 13 & 15 & 17 & \end{array} \right\},$$

hence $\mu^{(3)} = 16, 4, 2$. Note that no more double backward moves of the 3rd kind are possible without violating the conditions for $b_{4,3}(m, n)$. Since $r = 3 \not\equiv 3 - 1 = a - 1 \pmod 2$, we are in case (ii). 2 and 4 are 3-marked parts which appear an even number of times, and these are the 3rd and 2nd largest 3-marked parts, respectively. This means $j_3 = 2$, $n_3 = 3$, and the list $0 \leq 2 - 1 < 3 - 1 < n_r = 3$ will produce us the partition $\lambda_1 = 2 \times 1$, $\lambda_2 = 0$ using Lemma 3.4, indeed a partition enumerated by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_2$, the factor and the subscript 2 because of double moves instead of single ones. Along with q^2, the two single moves, we have the partition $\nu^{(3)} = 3, 1$. After performing the single backward moves of the 3-rd kind on 3-marked 2 and 3-marked 4, we perform the remaining double move on 3-marked 6, and transform λ further to

$$\left\{ \begin{array}{ccccccccccccc} & 2 & & 4 & & 6 & & & & & & & \\ 1 & & 3 & & 5 & & & 9 & & & 15 & & 18 \\ 1 & & 3 & & 5 & & 8 & & 11 & 13 & 15 & 17 & \end{array} \right\}.$$

Observe that these last backward moves are also accounted for by $y^{n_r}(\frac{1}{y})^{j_r} = y^3(\frac{1}{y})^2 = y$. Indeed, for parts $\geq$ the largest of the 3-marked parts, we still need to account for y^2, in accordance with the arguments above.

We now look at the 3 largest $r = 2$-marked parts 9, 15, and 18. 1 double backward move of the 2nd kind is possible on the 2-marked 9, followed by 3 double backward moves on 2-marked 15, which are in turn followed by 3 double backward moves of the 2nd kind on the 2-marked 18. Thus, $\mu^{(2)} = 6, 6, 2$, and the current state is

$$\left\{ \begin{array}{ccccccccccccc} & 2 & & 4 & & 6 & & & & & & & \\ 1 & & 3 & & 5 & & & 8 & 10 & & 13 & & \\ 1 & & 3 & & 5 & & 7 & & 10 & 12 & & 15 & 17 \end{array} \right\}.$$

The 1st and 3rd smallest 2-marked parts among the 3 largest correspond to even numbers that occur an even number of times. Since $r = 2 \leq 2 = a - 1$, we are in case (i). We need two single backward moves of the 2nd kind to convert the partition at hand to the base partition except for the largest 2 of the 1-marked ones, hence $j_2 = 2$. And single backward move of the 2nd kind on 2-marked 8 will grant the 2-marked parts 10 and 13 one double move each. Those single moves are accounted for by $(qy)^2$, and Lemma 3.4 from the list $0 \leq 1 - 1 < 3 - 1 < n_r = 3$ produces $\lambda_1 = 1$, $\lambda_2 = 1$, both referring to double moves. This is a partition generated by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_2$. Along with the single moves q^2, this yields $\nu^{(2)} = 3, 3$ $(3 \leq 2n_2 - 2j_2 + 1 = 6 - 4 + 1 = 3)$. Performing the described backward moves, we get

$$\left\{ \begin{array}{ccccccccccc} & 2 & & 4 & & 6 & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & 15 & 17 \end{array} \right\}.$$

Note that the remaining y^2 is already accounted for, so we do not expect any even parts larger than the largest of the 2-marked parts that appear an odd number of times.

Obviously, $j_1 = 0$ for the 2 largest 1-marked parts 15 and 17. So $\nu^{(1)}$ is the empty partition, realizing the assertion in the previous paragraph. a double backward move of the 1st kind on both brings $\mu^{(1)} = 2, 2$ and λ is eventually made into

$$\widetilde{\lambda} = \left\{ \begin{matrix} & 2 & & 4 & & 6 & & & & & & & & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & 15 & \end{matrix} \right\}.$$

For the other direction, given integers $n_1, \ldots, n_{k-1}$, we construct $\widetilde{\lambda}$ as in the proof of Theorem 3.1. For each $r = 1, 2, \ldots, k-1$ in increasing order, we consider the n_r largest of the r-marked parts $\lambda_{l_1} < \lambda_{l_2} < \cdots < \lambda_{l_{n_r}}$. Here, $\nu^{(r)}$ is a partition listed by $q^{j_r} \begin{bmatrix} n_r \\ j_r \end{bmatrix}_2$. The contribution from $\begin{bmatrix} n_r \\ j_r \end{bmatrix}$ fed into Lemma 3.4 will give us a list $0 \le i_1 - 1 < i_2 - 1 < \cdots < i_{j_r} - 1 < n_r$ weighted by the sum of the smallest elements in maximal sublists of consecutive integers. The j_r r-marked parts that will be applied the single forward moves of the rth kind are the i_1th, i_2th, …, i_{j_r}th largest r-marked parts. To realize those single moves, we take $\lambda_{l_1} < \lambda_{l_2} < \cdots < \lambda_{l_{n_r}}$ one by one in decreasing order.

For λ_{l_i}, if $i = i_s$ for some i_s in the constructed list, we perform a single forward move of the rth kind on λ_{l_i}. However, unless $i = 1$ or $i - 1 = i_{s-1}$ we cannot make a forward move of the rth kind on λ_{l_i} by construction of $\widetilde{\lambda}$. To make this move possible by Proposition 2.8(ii), we perform double moves on $\lambda_{l_1}, \lambda_{l_2}, \ldots, \lambda_{l_{i-1}}$ first, then make the (single) forward move of the rth kind on λ_{l_i}. The sequence of double moves ensure that the designated parts, and no other parts, become the even parts or succeed even parts the number of occurrences of which is of a fixed parity. This is the case if $i_s - 1$ is the smallest element of a maximal subsequence of consecutive integers in the above list, and so the required double moves are granted since the list is weighted by the sum of those smallest numbers.

For the assignment of y factors, there are two cases.

(i) $r > a - 1$ and $r \equiv a - 1 \pmod 2$, or $r \le a - 1$

In this case, $\nu^{(r)}$ is enumerated by $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (qy)^{j_r}$. So we assign y to each r-marked λ_{l_i} which was applied one of the single forward moves of the rth kind.

(ii) $r > a - 1$ and $r \not\equiv a - 1 \pmod 2$

In this case, $\widetilde{\lambda}$ had the n_r largest r-marked parts even numbers that appear an odd number of times. So, initially $\widetilde{\lambda}$ was accompanied by y^{n_r}. Also, $\nu^{(r)}$ is enumerated by $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (\frac{q}{y})^{j_r}$, and each single move rules out an even part that occur an odd number of times. Thus, λ_{l_i} that were applied single forward moves of the rth kind are assigned $\frac{1}{y}$.

Finally, we realize $\mu^{(r)}$ as in the proofs of Theorems 3.2 and 3.3. We apply $\frac{\mu_i^{(r)}}{2}$ double forward moves of the rth kind on the ith largest r-marked part for $i = 1, 2, \ldots, n_r$, the largest first.

By construction, the exponent of y is the number of even parts that appear an odd number of times at all times.

To conclude the proof, we recall that forward and backward moves on the same part are inverse of each other. Also, everything in the second construction is done in the exact reverse order of the first construction above, and vice versa. □

Example. For $k = 4$, $a = 2$, given

$$n_1 = 3, \quad n_2 = 4, \quad n_3 = 3,$$
$$\mu^{(1)} = \text{the empty partition}, \qquad \mu^{(2)} = 4, 2, \qquad \mu^{(3)} = 14, 10, 8,$$
$$\nu^{(1)} = 5, \qquad \nu^{(2)} = 5, 5, \qquad \nu^{(3)} = 3, 3,$$

we will construct a partition λ enumerated by $b_{4,2}(20, 227)$ accompanied by a power of y where the exponent counts the even parts that appear an odd number of times. We begin by constructing the base partition described in Theorem 3.1,

$$\widetilde{\lambda} = \left\{ \begin{matrix} & 2 & & 4 & & 6 & & & & & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & 14 & & & & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & 15 & & 17 & & 19 & \end{matrix} \right\}.$$

Note that by the generating function (3.5), we initially have an accompanying $y^{n_2} = y^4$, since 8, 10, 12, 14 are the even parts that appear an odd number of times. Looking at the number of parts in $\nu^{(r)}$, we expect λ to be accompanied by

$$y^5 = y^4 y^1 y^{-2} y^2,$$

where $4 = n_2$, and $1, 2, 2$ are the numbers of parts in $\nu^{(r)}$ in their respective order. 2 has a minus sign since $r = 2 \not\equiv 1 = a - 1 \pmod 2$.

We first apply $\nu^{(1)} = 5$. When we subtract 1 from all parts of $\nu^{(1)}$ (just one single move to be performed in this case), and divide by 2 (hence counting the double moves), we have the partition $2(\times 2)$ enumerated by $\begin{bmatrix} n_1 \\ j_1 \end{bmatrix}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_2$. Lemma 3.4 produces the list $\{3 - 1\}$. This means that we need to apply a single move on the 3rd largest 1-marked part 15. And to enable that move without applying any other single move, we need to perform double forward moves of the 1st kind on the 1-marked 21 and 1-marked 19. These moves are granted by the partition $2(\times 2)$ mentioned above. At this point, we have

$$\left\{\begin{array}{ccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & 14 & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & 16 & 19 & 21 \end{array}\right\}.$$

We have another y here thanks to the single move we made. With the initial y^4, that makes y^5 We now have another even part (16) that appears an odd number of times. $\mu^{(1)}$ does not change anything because it is the empty partition.

Then, for $r = 2$, we subtract 1 from all parts of $\nu^{(2)}$ (2 single moves to be made) and feed the remaining partition in Lemma 3.4 after dividing the remaining even parts (4, 4) by 2 to get the list $\{2 - 1, 4 - 1\}$ from the partition 2, 2 enumerated by $\begin{bmatrix} n_2 \\ j_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Note that 2, 2 denotes double moves. Both elements in the list are smallest elements in maximal sublists, so the list is weighted by the sum of its elements in this case. Indeed, we need to apply single forward moves of the 2nd kind on the 2nd and 4th largest 2-marked elements. Thus, we first need to perform a double move on the largest 2-marked part before applying the first single move, and a double move on each 1st, 2nd and 3rd largest 2-marked parts before applying the second single move.

We already see that the 2-marked 12 cannot be moved forward since both (d1) and (d2) fail in Definition 2.6. We show the process step by step.

After a double forward move performed on the largest 2-marked part, we have

$$\left\{\begin{array}{ccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & 15 & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 14 & & 16 & 19 & 21 \end{array}\right\}.$$

This, after a single forward move performed on the 2nd largest 2-marked part, becomes

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 10 & 12 & & 15 & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 12 & 14 & & 16 & 19 & 21 \end{array}\right\}.$$

After double forward moves performed on the 3 largest 2-marked parts, it becomes

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 11 & & 14 & & 17 & & \\ 1 & & 3 & & 5 & & 7 & & 10 & & 12 & 14 & 16 & & 19 & 21 \end{array}\right\}.$$

And, after a single forward move performed on the 4th largest 2-marked part, we get

$$\left\{\begin{array}{ccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & \\ & 2 & & 4 & & 6 & 8 & & 11 & & 14 & & 17 & & \\ 1 & & 3 & & 5 & & 8 & 10 & & 12 & 14 & 16 & & 19 & 21 \end{array}\right\}.$$

Note that single moves rule out even parts that originally occurred an odd number of times, hence we have y^{-2} at this stage. With current y^5, this yields y^3. Indeed, there are 3 even parts (10, 12, and

16) that appear an odd number of times. Now we apply $\mu^{(2)}$ as described in the example following Theorem 3.3 to obtain

$$\left\{\begin{array}{ccccccccccccccc}
 & 2 & & 4 & & 6 & & & & & & & & & \\
 & 2 & & 4 & & 6 & 8 & & 11 & & 15 & & & 20 & \\
1 & & 3 & & 5 & & 8 & 10 & & 12 & 15 & 17 & 19 & & 21
\end{array}\right\}.$$

For $r = 3$, we proceed as in the previous paragraphs. We subtract 1 from parts of $\nu^{(3)}$ (which accounts for the single moves to be performed), and divide parts of the remaining partition by 2 (hence count the double moves) to get 1, 1. Lemma 3.4 produces the list $\{1-1, 3-1\}$ with weight $0+2=2$ (both elements are smallest in their respective maximal sublist). Therefore, we need to perform single forward moves of the 3rd kind on the 1st and 3rd largest 3-marked parts. In order to do that, two double forward moves are needed, one on each of the largest two 3-marked parts. To be more precise, the single forward move of the 3rd kind on the largest 3-marked part is possible regardless; however, for the single move on the 3rd largest 3-marked part, we first need to move the larger 3-marked parts forward. After the single move of the 3rd kind performed on the largest 3-marked part 6, we have

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & 4 & 6 & & & & & & & & & \\
 & 2 & & 4 & 6 & 8 & & 11 & & 15 & & & 20 & \\
1 & & 3 & & 6 & 8 & 10 & & 12 & 15 & 17 & 19 & & 21
\end{array}\right\}.$$

This, after double moves of the 3rd kind performed on the 1st and 2nd largest 3-marked parts followed by the single move of 3rd kind on the 3-marked 2, becomes

$$\left\{\begin{array}{ccccccccccccc}
2 & & 5 & & 8 & & & & & & & & \\
2 & 4 & & 6 & 8 & & 11 & & 15 & & & 20 & \\
2 & 4 & & 6 & 8 & 10 & & 12 & 15 & 17 & 19 & & 21
\end{array}\right\}.$$

The required double forward moves are granted by the weight of the list. We have y^2 here, since we introduced two even parts (2 and 8) that now occur an odd number of times. Together with y^3 from the previous run, we now have y^5 (2, 4, 10, 12, and 20 are the even parts that appear an odd number of times).

Finally, we incorporate $\mu^{(3)}$ to get the final partition λ.

$$\lambda = \left\{\begin{array}{ccccccccccccc}
 & & & & 8 & 11 & & & & 17 & & & \\
2 & & 5 & & 8 & 11 & 13 & & 16 & & & 20 & \\
2 & 4 & & 6 & 8 & 11 & 13 & 15 & & 17 & 19 & & 21
\end{array}\right\},$$

along with y^5, as asserted.

4. Conclusion and further research

The main difference from Bressoud's approach in [9, Section 5] is that in Section 3, the construction is direct instead of inductive. For $k = 2$, both constructions are clearly the same. For larger k, the empirical evidence is strong that both methods produce the same partitions with the same inputs. Bressoud's method also seems to keep the mentioned attributes invariant for the case of Andrews–Gordon identities. Therefore, a further research problem is to prove (or to disprove, which does not seem likely) the equivalence of both methods in explaining the series side of the Andrews–Gordon Identities. Another research problem is to modify Bressoud's inductive argument so that it proves Theorem 3.5.

On the other hand, it is not possible to explain Bressoud's generalization [9, Eq. (5.3)] of (1.4) using Gordon marking and backward and forward moves. To be more precise, in [9, Eq. (5.3)], Bressoud keeps track of the violation of a divisibility condition for parts. It is straightforward to produce instances where the mentioned violation of that divisibility condition implies violation of Gordon marking as well, when one sticks to backward and forward moves.

The other approaches to interpret (1.4) have some fundamental differences from the method discussed here. For example, in [4], the general term of (1.4) is interpreted after some modification, so that a part may well appear more than k times. In [11], the approach is again inductive. Furthermore, the way lattice paths are weighted only allows distinct non-consecutive integers as weights of separate peaks in a lattice path. This means, if a direct bijection exists between the lattice paths described in [11] and the partitions enumerated by $b_{k,a}(m, n)$, it has to be highly non-trivial. A direct bijection is meant as opposed to unfolding one way, and constructing the other way. The discovery of such a correspondence may be another research problem.

In [8], Bressoud gave a generalization of the Rogers–Ramanujan Identities for all moduli. He introduced $B_{k,a,\delta}(n)$, the number of partitions of n such that

$$f_1 \le a-1, \quad f_i + f_{i+1} \le k-1, \quad \delta = 0 \quad \text{or} \quad 1,$$
$$\text{and if } f_i + f_{i+1} = k-1, \text{ then } if_i + (i+1)f_{i+1} \equiv a-1 (\text{mod } 2-\delta);$$

where f_i is the number of occurrences or the frequency of i in a given partition. In [10], he gave the following generating function for $B_{k,a,\delta}(m, n)$, the number of partitions enumerated by $B_{k,a,\delta}(n)$ which have exactly m parts. Note that this is a subset of partitions enumerated by $b_{k,a}(m, n)$.

$$\sum_{m,n\ge 0} B_{k,a,\delta}(m, n) x^m q^n = \sum_{n_1,\ldots,n_{k-1}\ge 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}} x^{N_1+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-2}}(q^{2-\delta}; q^{2-\delta})_{n_{k-1}}}. \tag{4.1}$$

In the above equation, $N_i = n_i + \cdots + n_{k-1}$. Gordon marking and backward and forward moves explain the generating function (4.1). The proof of Theorem 3.1 carries over. One only needs an extra observation about the $(k-1)$-marked parts and double moves of the $(k-1)$th kind when $\delta = 0$.

The close inspection of Theorem 3.5 along with its proof makes one naturally conjecture and easily prove that the following functions generate partitions enumerated by $b_{k,a}(m, n)$, where the exponent of q is the number being partitioned, the exponent of x accounts for the number of parts, and the exponent of y keeps track of the number of *odd* parts that appear an odd number of times.

$$\sum_{n_1,\ldots,n_{k-1}\ge 0} q^{N_1^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}} \quad x^{N_1+\cdots+N_{k-1}} \quad y^{n_1+n_3+\cdots+n_{a-1}+n_a+n_{a+1}\cdots+n_{k-1}} \quad \cdots$$
$$\times \frac{\widetilde{H}_{n_1} H_{n_2} \widetilde{H}_{n_3} \cdots H_{n_{a-2}} \widetilde{H}_{n_{a-1}} \widetilde{H}_{n_a} \widetilde{H}_{n_{a+1}} \cdots \widetilde{H}_{n_{k-1}}}{(q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{k-1}}} \quad (\text{for } a \text{ even}) \tag{4.2}$$

$$\sum_{n_1,\ldots,n_{k-1}\ge 0} q^{N_1^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}} \quad x^{N_1+\cdots+N_{k-1}} \quad y^{n_1+n_3+\cdots+n_{a-2}} \quad \cdots$$
$$\times \frac{\widetilde{H}_{n_1} H_{n_2} \widetilde{H}_{n_3} \cdots \widetilde{H}_{n_{a-2}} H_{n_{a-1}} H_{n_a} H_{n_{a+1}} \cdots H_{n_{k-1}}}{(q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{k-1}}} \quad (\text{for } a \text{ odd}) \tag{4.3}$$

Then, one could set up a refinement of [2, Eq. (2.2)]. This will be a multiple recurrence for the functions (3.5), (4.2), and (4.3), also involving the extra parameter y. It is straightforward to check analytically that the recurrence is satisfied. Once one provides suitable initial conditions, this would supply the analytic proof of Andrews' Conjecture [7, Section 5], with a slight generalization.

The real challenge in the analytic approach is to find generalizations of the function $Q_{k,a}(x; q)$ (1.2) involving y, which would give another set of solutions to the multiple recurrence described in the above paragraph. This would not only give a new class of identities, but would also unify all results in [7, Section 2–5]. A good starting point might be [3].

For a more extensive list of related open problems, the reader is referred to [7, Section 13].

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